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LETTER TO THE EDITOR

Classification of the extended symmetries of Fokker–Planck equations

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Abstract. We obtain a complete classification of the possible types of the extended symmetry of any Fokker-Planck equation, together with a necessary and sufficient condition for each type. We give also the most general form (up to a change of variables) of the Lie generators of these symmetries, and a comparison with the case of the heat equation.

Extended symmetry properties of the heat equation, and more generally of Fokker-Planck-type equations, which we will write here in the form ($t \in \mathbb{R}$, $x \in \mathbb{R}$, $f = f(x, t)$)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(a(x)f) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2(x)f) \tag{1}$$

have been the subject of several papers (see e.g. [1-5] and references therein). Referring to [6, 7] and the above quoted papers for further details, let us recall that the main purpose is to find the maximal group of local (continuous) symmetries of the equations and more precisely their Lie generators, which can be written in the general form

$$v = \xi(x, t, f)\frac{\partial}{\partial x} + \tau(x, t, f)\frac{\partial}{\partial t} + \phi(x, t, f)\frac{\partial}{\partial f} \tag{2}$$

where ξ, τ, ϕ are functions to be determined. In a recent paper [4], we provided a general condition on the functions $a(x)$, $g(x)$ appearing in (1) in order that some non-trivial symmetry exists. In the present letter, we propose a natural classification of the possible symmetries admitted by the Fokker-Planck equations (1). Precisely, we obtain the three following cases.

(I) The symmetry group is (locally) isomorphic to the group of the heat equation; that is, there are *four* 'non-trivial' symmetry generators. This case happens if and only if there is a change of variables (x, t, f), not perturbing symmetry, which transforms (1) into the heat equation; this situation has been considered in [8] (see also [3, 5]).

(II) There are *two* non-trivial symmetry generators, leading to a symmetry subgroup of case I.

(III) No non-trivial symmetry is allowed.

We will give also the necessary and sufficient condition for each case, and the general form (up to a change of variables) of the symmetry generators.

Let us recall that it is easy to show [4, 7] that the coefficient ϕ in (2) has the general form

$$\phi = \alpha(x, t) + \beta(x, t) f \tag{3}$$

where $\alpha(x, t)$ is any solution of (1), and that the case

$$\tau = c_1 \qquad \beta = c_2 \tag{4}$$

where c_1, c_2 are constants, gives rise to the 'trivial' symmetries generated by

$$v' = \frac{\partial}{\partial t} \qquad v'' = f\frac{\partial}{\partial f} \qquad v''' = \alpha\frac{\partial}{\partial f} \tag{5}$$

which correspond precisely to the properties of (1) of being autonomous and linear. So we will be exclusively interested, from now on, in 'non-trivial' symmetries. First of all, let us assume in (1)

$$g = \text{constant} \neq 0.$$

This is not a restriction; in fact this situation can be achieved by means of a change of variable [9] not altering the symmetry properties of the original equation. Using then our method [4], which in turn is based on standard Lie–Olver procedure [6, 7], we obtain in this case (here and in the following subscripts mean differentiation)

$$\xi(x, t) = c(t) g + \frac{x}{2}\tau_t$$

where $c = c(t)$ is a function to be determined, and the necessary and sufficient condition for the existence of some symmetry is (equation (14) of [4])

$$c_{tt} + M_x c = -(2g)^{-1}(x\tau_{ttt} + xM_x\tau_t + 3M\tau_t) \tag{6}$$

where

$$M = -\tfrac{1}{2}(a^2 + g^2 a_x)_x. \tag{6'}$$

In order to fulfil condition (6), assume now the following.

(I) $a(x)$ satisfies the equation

$$M_{xx} = 0 \tag{7}$$

i.e.

$$a^2 + g^2 a_x = \mu x^2 + \mu' x + \mu_0 \tag{7'}$$

where μ, μ', μ_0 are constants. We will give at the end a class of solutions of this equation. Then, differentiating (6) with respect to x, we get

$$\tau_{ttt} - 4\mu\tau_t = 0 \qquad c_{tt} - \mu c = \frac{3\mu'}{4g}\tau_t. \tag{8}$$

Using again a procedure given in [4], we are now able to evaluate $\beta(x, t)$ and then obtain the following four symmetry generators:

$$v_1 = \frac{e^{2\sqrt{\mu}t}}{2\sqrt{\mu}}\frac{\partial}{\partial t} + \frac{1}{2}\left(x + \frac{\mu'}{2\mu}\right)e^{2\sqrt{\mu}t}\frac{\partial}{\partial x}$$
$$+ \frac{e^{2\sqrt{\mu}t}}{2g^2}\left[a(x)\left(x + \frac{\mu'}{2\mu}\right) - \sqrt{\mu}x^2 - \frac{\mu'}{\sqrt{\mu}}x - \frac{\mu_0}{2\sqrt{\mu}} - \frac{(\mu')^2}{8\mu\sqrt{\mu}} - \frac{g^2}{2}\right] f\frac{\partial}{\partial f}$$

$$v_2=-\frac{e^{-2\sqrt{\mu}t}}{2\sqrt{\mu}}\frac{\partial}{\partial t}+\frac{1}{2}\left(x+\frac{\mu'}{2\mu}\right)e^{-2\sqrt{\mu}t}\frac{\partial}{\partial x}$$
$$+\frac{e^{-2\sqrt{\mu}t}}{2g^2}\left[a(x)\left(x+\frac{\mu'}{2\mu}\right)+\sqrt{\mu}x^2+\frac{\mu'}{\sqrt{\mu}}x+\frac{\mu_0}{2\sqrt{\mu}}+\frac{(\mu')^2}{8\mu\sqrt{\mu}}-\frac{g^2}{2}\right]f\frac{\partial}{\partial f} \qquad (9)$$
$$v_3=g\,e^{\sqrt{\mu}t}\frac{\partial}{\partial x}+\frac{e^{\sqrt{\mu}t}}{g}\left[a(x)-\sqrt{\mu}\left(\frac{\mu'}{2\mu}+x\right)\right]f\frac{\partial}{\partial f}$$
$$v_4=g\,e^{-\sqrt{\mu}t}\frac{\partial}{\partial x}+\frac{e^{-\sqrt{\mu}t}}{g}\left[a(x)+\sqrt{\mu}\left(\frac{\mu'}{2\mu}+x\right)\right]f\frac{\partial}{\partial f}.$$

Note that the constants μ, μ', μ_0 may be zero (the modifications needed in (9) if $\mu=0$ can be easily performed starting from (8)). This is then the *most general form* of the symmetry generators in case I (up to a change of variables in the case g not constant) for the Fokker-Planck equation (1).

It is interesting to remark that condition (7) is actually the necessary and sufficient condition in order that Fokker-Planck equation (1) can be transformed by means of a suitable change of variables, not altering symmetry, into the heat equation [5, 8]. Therefore, taking into account also the foregoing analysis of the cases $M_{xx}\neq 0$, we can directly see that the only way for a Fokker-Planck equation to possess four non-trivial symmetry generators is for it to be transformed into the heat equation. In this case, the symmetry group is then locally isomorphic to the group of the heat equation, which is obtained with $\mu=\mu'=\mu_0=0$. Another way to state the above property is the following: equation (1) can be also transformed into a Schrödinger equation with potential $V(x)=a^2+g^2a_x$ [9]. Then, condition (7) allows precisely a polynomial, at most quadratic, potential. But it is also known, rather surprisingly [10, 11], that this Schrödinger equation has the same extended group (up to local isomorphism) as the free equation, with $V(x)=0$.

(II) Let us assume now $M_{xx}\neq 0$. Some straightforward calculations based on (6) and (6′) show that, if there are two constants ν, ν_0 such that the following equation for $M(x)$ is satisfied

$$(M_x+\nu)(x+\nu_0)+3M=0 \qquad (10)$$

then condition (6) can be fulfilled if and only if

$$\tau_{ttt}-\nu\tau_t=0 \qquad c=\frac{\nu_0}{2g}\tau_t. \qquad (11)$$

When written in terms of $a(x)$, condition (10) reads

$$a^2+g^2a_x=\frac{\nu}{4}(x+\nu_0)^2+\frac{\nu_1}{(x+\nu_0)^2}+\nu_2 \qquad (10')$$

where ν_1, ν_2 are constants. This situation gives rise to only two non-trivial symmetry generators, given by

$$v_1=\frac{e^{\sqrt{\nu}t}}{\sqrt{\nu}}\frac{\partial}{\partial t}+\frac{1}{2}(x+\nu_0)\,e^{\sqrt{\nu}t}\frac{\partial}{\partial x}+\frac{e^{\sqrt{\nu}t}}{2g^2}\left(a(x)(x+\nu_0)-\frac{\sqrt{\nu}}{2}(x+\nu_0)^2+\frac{\nu_2}{\sqrt{\nu}}-\frac{g^2}{2}\right)f\frac{\partial}{\partial f}$$
$$v_2=-\frac{e^{-\sqrt{\nu}t}}{\sqrt{\nu}}\frac{\partial}{\partial t}+\frac{1}{2}(x+\nu_0)\,e^{-\sqrt{\nu}t}\frac{\partial}{\partial x} \qquad (12)$$
$$+\frac{e^{-\sqrt{\nu}t}}{2g^2}\left(a(x)(x+\nu_0)+\frac{\sqrt{\nu}}{2}(x+\nu_0)^2+\frac{\nu_2}{\sqrt{\nu}}-\frac{g^2}{2}\right)f\frac{\partial}{\partial f}.$$

Note in particular that, if $a(x)$ satisfies case I, then it is always possible to find a real number ν such that $a(x)$ satisfies also condition (10) of case II; therefore the symmetry described by II is merely a subgroup of that of the case I.

(III) If finally $a(x)$ is such that neither (7), (7′) nor (10), (10′) are satisfied, then the only allowed solution of (6) is

$$\tau_t = c = 0$$

which is the case of non non-trivial symmetry.

Interesting examples for each one of the three cases above can be obtained by considering the following choice (γ, λ, δ_1, δ_2 are constants):

$$a(x) = \gamma x + \frac{\lambda}{x} + \delta_1 \qquad g^2 = \lambda + \delta_2. \tag{13}$$

If $\lambda = 0$ or $\delta_1 = \delta_2 = 0$, then condition (7), (7′) is satisfied and case I is recovered. Let $\lambda \neq 0$; then if $\delta_1 = 0$ but $\delta_2 \neq 0$, condition (10), (10′) of case II holds true; if finally $\delta_1 \neq 0$, neither (7), (7′) nor (10), (10′) are satisfied, and the symmetry is completely removed. Then δ_1 and δ_2 acquire the clear interpetation of 'symmetry-breaking parameters'. Note that the choice (13) includes the so-called 'linear' Fokker–Planck equation ($\lambda = \delta_1 = 0$, and the heat equation as well), and equations for Rayleigh-type processes ($\lambda \neq 0$, $\delta_1 = 0$, $\delta_2 = 0$ or $\delta_2 = \lambda$).

It can be useful to remark finally that all above calculations greatly simplify in the special (but rather common) case in which $a(x)$ is an odd function (cf [1]). Then condition (6) immediately splits into two independent conditions involving $c(t)$ and $\tau(t)$ separately, and the discussion of the three cases goes very easily.

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